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THE ARF AND SATO LINK CONCORDANCE INVARIANTS

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ABSTRACT. The Kervaire-Arf invariant is a $Z/2$ valued concordance invariant of knots and proper links. The β invariant (or Sato's invariant) is a Z valued concordance invariant of two component links of linking number zero discovered by J. Levine and studied by Sato, Cochran, and Daniel Ruberman. Cochran has found a sequence of invariants $\{\beta_i\}$ associated with a two component link of linking number zero where each β_i is a Z valued concordance invariant and $\beta_0 = \beta$. In this paper we demonstrate a formula for the Arf invariant of a two component link $L = X \cup Y$ of linking number zero in terms of the β invariant of the link:

$$\operatorname{arf}(X \cup Y) = \operatorname{arf}(X) + \operatorname{arf}(Y) + \beta(X \cup Y) \pmod 2.$$

This leads to the result that the Arf invariant of a link of linking number zero is independent of the orientation of the link's components. We then establish a formula for $|\beta|$ in terms of the link's Alexander polynomial $\Delta(x, y) = (x-1)(y-1)f(x, y)$:

$$|\beta(L)| = |f(1, 1)|.$$

Finally we find a relationship between the β_i invariants and linking numbers of lifts of X and Y in a $Z/2$ cover of the compliment of $X \cup Y$.

1. INTRODUCTION

The Kervaire-Arf invariant [KM, R] is a $Z/2$ valued concordance invariant of knots and proper links. The β invariant (or Sato's invariant) is a Z valued concordance invariant of two component links of linking number zero discovered by Levine (unpublished) and studied by Sato [S], Cochran [C], and Daniel Ruberman. Cochran [C] has found a sequence of invariants $\{\beta_i\}$ associated with a two component link of linking number zero where each β_i is a Z valued concordance invariant and $\beta_0 = \beta$. 23 Theorem 1 demonstrates a formula for the Arf invariant of a two component link $L = X \cup Y$ of linking number zero in terms of the β invariant of the link:

Theorem 1. $\operatorname{arf}(X \cup Y) = \operatorname{arf}(X) + \operatorname{arf}(Y) + \beta(X \cup Y) \pmod 2$.

This leads to the result that the Arf invariant of a link of linking number zero is independent of the orientation of the link's components. In Theorem 2 we establish a formula for $|\beta|$ in terms of the link's Alexander polynomial $\Delta(x, y) = (x-1)(y-1)f(x, y)$.

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Theorem 2. $|\beta(L)| = |f(1, 1)|$.

Finally, in Theorem 3 we find a relationship between the invariants and linking numbers of lifts of X and Y in a $Z/2$ cover of the compliment of $X \cup Y$:

Theorem 3. $-\frac{1}{2}\operatorname{lk}(Y^0, Y^1) = \sum_{j=0}^{N-1} 4^j \beta_j(L) - 2(4)^{N-1}\operatorname{lk}(c_N^0, c_N^1)$.

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We will begin by recalling a few definitions.

Let V be a vector space of dimension $2n$ over $Z/2$. Let $\varphi: V \times V \to Z/2$ be a nonsingular symmetric bilinear form such that $\varphi(a, a) = 0$ for every a in V. The map $q: V \to Z/2$ is a quadratic form with respect to φ if and only if

$$q(a+b) = q(a) + q(b) + \varphi(a, b) \quad \text{for every } a, b \text{ in } V.$$

Definition 1.1. The set $\{a_1, b_1, \ldots, a_n, b_n\}$ is a symplectic basis of V with respect to φ if it is a basis of V and if $\varphi(a_i, b_j) = \delta_{ij}$ and $\varphi(a_i, a_j)$ $\varphi(b_i, b_j) = 0$.

Definition 1.2 (Arf invariant of a quadratic form). Let $\{a_1, b_1, \ldots, a_n, b_n\}$ be a symplectic basis of V with respect to φ. Then

$$\operatorname{arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i)$$

is the Arf invariant of q.

The above definition is independent of the choice of symplectic basis (see Arf [A]).

Let K be a knot in S^3 and M an oriented Seifert surface spanning K. Then $H_1(M; Z/2)$ is a vector field over $Z/2$ of dimension $2q$ where $q =$ genus of M. We have a symmetric nonsingular bilinear form

$$\operatorname{int}_2: H_1(M; Z/2) \times H_1(M; Z/2) \to Z/2$$

defined by $\operatorname{int}_2(\underline{a}, \underline{b}) =$ the mod 2 intersection number of cycles a, b which represent $\underline{a}$ and $\underline{b}$. Since M is orientable, $\operatorname{int}_2(\underline{a}, \underline{a}) = 0$ for every $\underline{a}$ in $H_1(M; Z/2)$.

From now on we will assume we have fixed an orientation of S^3. Since M and S^3 are now oriented we can distinguish between the positive and negative normal directions to M.

Given $\underline{a}$, an element of $H_1(M)$, let a be a curve in M which represents $\underline{a}$ and a^+, a^- the push offs of a in the positive and negative normal directions to M. Define $\delta_2: H_1(M; Z/2) \to Z/2$ by $\delta_2(\underline{a}) = \operatorname{lk}(a, a^+)$ mod 2 the modulo two linking number between a and a^+. We have

$$\delta_2(\underline{a}+\underline{b}) = \delta_2(\underline{a}) + \delta_2(\underline{b}) + \mathrm{int}_2(\underline{a}, \underline{b})$$

[KM], so δ_2 is a quadratic form associated with int_2.

Definition 1.3 (Arf invariant of a knot). The Arf invariant of a knot K is equal to the Arf invariant of the quadratic form δ_2.

Let $L = X \cup Y$ be a two component link with even linking number in S^3 whose components X and Y are oriented. Let M be a connected Seifert surface for L. Then $H_1(M; Z/2)$ is a vector space over $Z/2$ of dimension $2g+1$ where $g =$ genus of M. The modulo two intersection form is nonsingular on the quotient space of $H_1(M; Z/2)$ given by $H_1(M; Z/2)/J$, where J is a one-dimensional subspace generated by one boundary component of M.

Definition 1.4 (Arf invariant of a link). The Arf invariant of link L is equal to the Arf invariant of the quadratic form δ_2 restricted to the quotient space $H_1(M; Z/2)/J$ of $H_1(M; Z/2)$.

The above definition is well defined for links of even linking number [R].

If $L = X \cup Y$ has linking number zero then there exist Seifert surfaces M_x, M_y for X and Y such that $X \cap M_Y = Y \cap M_X = \varnothing$. Therefore $M_x \cap M_y = C_1 \cup C_2 \cup \cdots \cup C_n$ where C_j is a circle embedded in M_x and M_y. Orient C_j so that a positive unit tangent to C_j together with the positive unit normals to M_x and M_y give the chosen orientation of S^3.

The intersection $M_x \cap M_y$ has a natural framing of its normal bundle given by the normal 1-fields $(\vec{v}, \vec{w})$. By the Thom-Pontryagin construction $(M_x \cap M_y, \vec{v}, \vec{w})$ corresponds to an element of $\Pi_3(S^2)$.

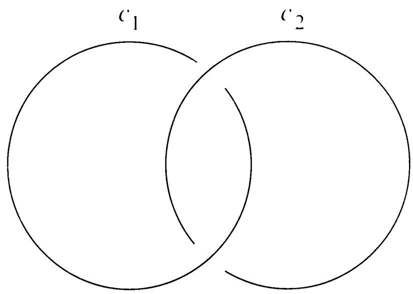
Definition 1.5 (the β or Sato invariant). $\beta(L)$ is the element of $\Pi_3(S^2)$ given by $(M_x \cap M_y, \vec{v}, \vec{w})$.

Alternate definition to 1.5a (the β or Sato invariant).

$$\beta(L) = 2\sum_{i<j} \mathrm{lk}(C_i, C_j) + \sum_{j=1}^{n} \mathrm{lk}(C_j, C_j^+)$$

where $\mathrm{lk}(a, b)$ is the linking number between a and b with respect to the chosen orientation of S^3.

Definitions 1.5 and 1.5a are obviously the same if $M_x \cap M_y$ is connected. In Lemma 2.1 we show that if $M_x \cap M_y$ has two or more components then M_x and M_y can be altered so that the altered surfaces intersect in a connected manifold. Figure 1.1 illustrates how the definitions are the same when $M_x \cap M_y = c_1 \cup c_2$ has two components and M_x, M_y are altered so that the new intersection c is connected.



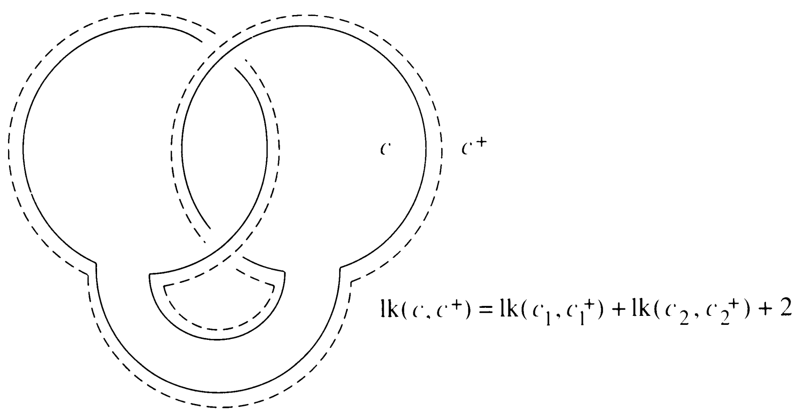


FIGURE 1.1

2. A RELATIONSHIP BETWEEN THE β AND ARF INVARIANTS

In this section we establish a relationship between the β invariant and the Arf invariant of a link of linking number zero.

Theorem 1. *If $L = X \cup Y$ is an oriented link of linking number zero then* $\mathrm{arf}(L) = \mathrm{arf}(X) + \mathrm{arf}(Y) + \beta(L) \pmod 2$.

First we will find two oriented Seifert surfaces M_X, M_Y for X, Y respectively such that $M_X \cap M_Y$ consists of only one circle. We cut M_X and M_Y along this circle and reglue to obtain an oriented Seifert surface M for L. Surface M will be used to compute $\mathrm{arf}(L)$.

Lemma 2.1. *Let $L = X \cup Y$ be a link of linking number zero. There exist orientable Seifert surfaces M_X and M_Y spanning X and Y such that $M_X \cap M_Y = \varnothing$ or one circle. Furthermore this circle is not null homologous in $H_1(M_X)$ of $H_1(M_Y)$.*

Proof. Let N_X, N_Y be oriented Seifert surfaces for X and Y such that $N_X \cap N_Y = \{c_1, \ldots, c_n\}$, where each c_j is an embedded circle. Change circle inter-

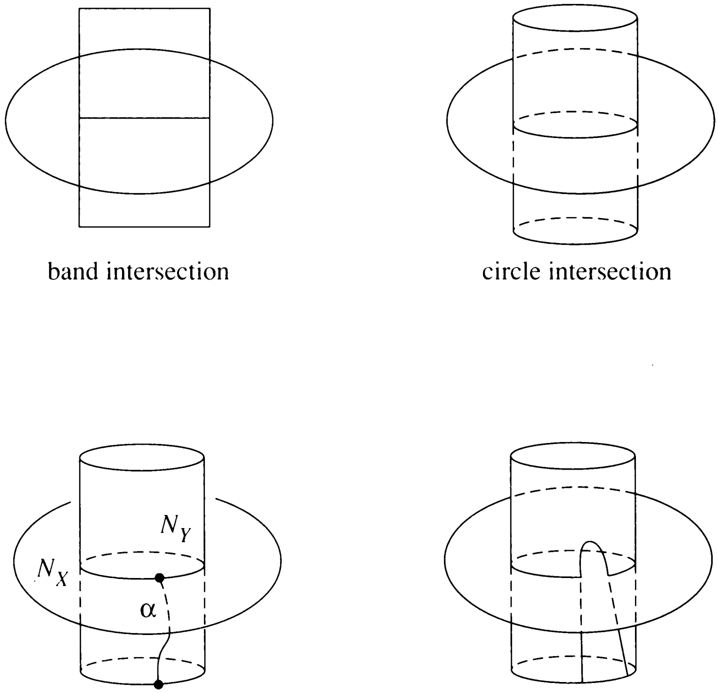


FIGURE 2.1

sections c_1 and c_2 to band intersections by the following method of Cooper [Co]. Let α_1, α_2 be paths from ∂N_Y to points on c_1 and c_2 respectively. Push N_Y in along α_1 and α_2 so c_1 and c_2 become band intersections (see Figure 2.1). Let p_1 and p_2 be two points on the resulting band intersections. Let γ be an arc in N_X from p_1 to p_2 and oriented from p_1 to p_2 such that the intersection number of γ with N_Y at p_1 is $+1$ and the intersection number of γ with N_X at p_2 is -1 (see Figure 2.2(i)). Now attach a handle to N_Y whose core coincides with γ (see Figure 2.2(ii)). Our two band intersections have been transformed into two different band intersections. If we attach two handles to N_X whose cores coincide with the two arcs in $\partial N_Y = Y$ which were pushed in along paths α_1 and α_2 then the band intersections will become one circle intersection (see Figure 2.2(iii)).

Let us call our new surfaces N'_X and N'_Y. These new surfaces intersect in $N'_X \cap N'_Y = \{a, c_3, \dots, c_n\}$, where a was formed from c_1 and c_2. If we repeat the above process we can find oriented surfaces M_X and M_Y such that $M_X \cap M_Y =$ one circle c.

Suppose $M_X \cap M_Y = c$ and c is null homologous in $H_1(M_X)$. Then c bounds a surface N in M_X. Look at a neighborhood of N in S^3 parametrized by $N \times [1, -1]$, where $N \times 0$ coincides with N. Replace M_X with $M_X - N \cup \partial N \times [1, 0] \cup N \times 1$. Now round off the corner at $N \times 0$ and we will no longer have an intersection along c. □

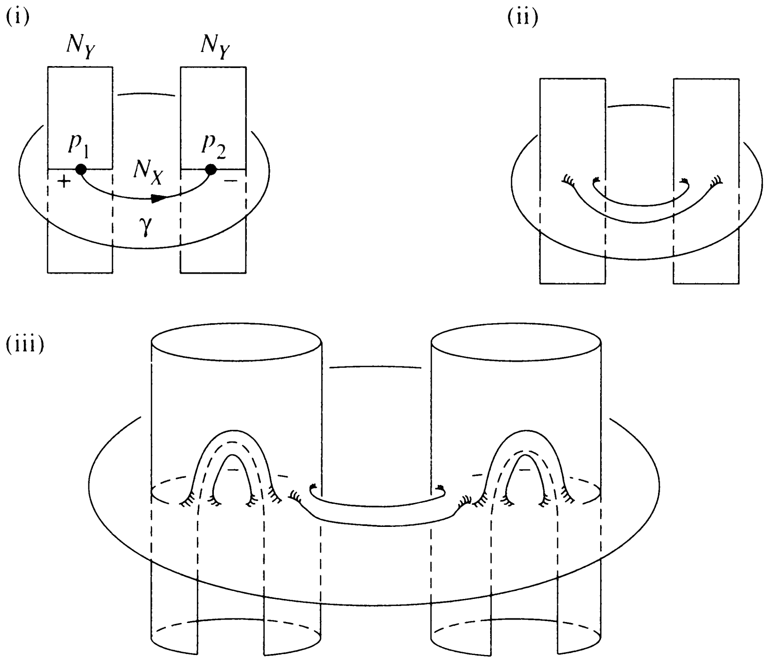


FIGURE 2.2

Proof of Theorem 1. By Lemma 2.1 we can assume we have oriented Seifert surfaces M_X and M_Y for X, Y respectively such that $M_X \cap M_Y = \varnothing$ or $M_X \cap M_Y = c$, where c is an embedded circle and c not homologous to zero in $H_1(M_X)$ or $H_1(M_Y)$.

If $M_X \cap M_Y = \varnothing$ then L is a boundary link. From the methods of Robertello [R] it easily follows that $\operatorname{arf}(L) = \operatorname{arf}(X) + \operatorname{arf}(Y)$. Since the β invariant for a boundary link is zero the formula holds in this case.

Assume that $M_X \cap M_Y = c$. We can find symplectic bases :

$$\{\underline{c}, \underline{d}_1, \underline{c}_2, \underline{d}_2, \dots, \underline{c}_g, \underline{d}_g\}, \quad \{\underline{c}, \underline{d}'_1, \underline{e}_2, \underline{f}_2, \dots, \underline{e}_h, \underline{f}_h\}$$

for $H_1(M_X; Z/2)$ and $H_1(M_Y; Z/2)$ respectively, where $g =$ genus M_X and $h =$ genus M_Y, and c represents $\underline{c}$.

We now construct M, an oriented Seifert surface for L, from M_X and M_Y. Cut M_X and M_Y along c. Attach $M_X - c$ to $M_Y - c$ so that the positive sides of $M_X - c$ and $M_Y - c$ form the positive side of M. M is orientable. Let $H_1(M; Z/2) = I \oplus J$, where $J =$ the null space of form int_2 and I is the subspace with symplectic basis :

$$\{\underline{c}, \underline{d}, \underline{c}_2, \underline{d}_2, \dots, \underline{c}_g, \underline{d}_g, \underline{e}_2, \underline{f}_2, \dots, \underline{e}_h, \underline{f}_h\}$$

where $d = d_1 \cup d'_1$ represents $\underline{d}$. The self-linking numbers of the basis element of I have not changed with the exception of $\underline{d}$: $\delta_2(\underline{d}) = \delta_2(\underline{d}_1) + \delta_2(\underline{d}'_1) + 1$.

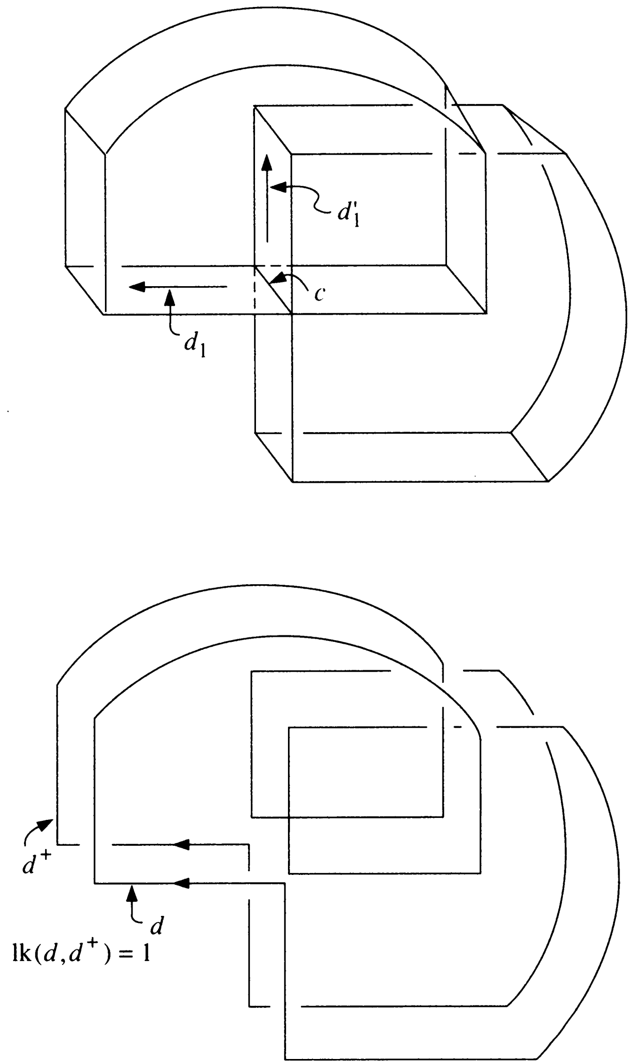


FIGURE 2.3

The extra 1 appears because $\underline{d}_1$ and $\underline{d}_1'$ intersects at a point. Figure 2.3 is a local picture of $M_X \cup M_Y$ near d_1 and d_1' and shows how the extra linking number appears.

Use the above symplectic basis of I to calculate $\mathrm{arf}(L)$:

$$\begin{aligned}\mathrm{arf}(L) &\equiv \delta_2(\underline{c})\delta_2(\underline{d}) + \sum_{j=2}^{g} \delta_2(c_j)\delta_2(\underline{d}_j) + \sum_{j=2}^{h} \delta_2(\underline{e}_j)\delta_2(\underline{f}_j) \\ &\equiv \delta_2(\underline{c})(\delta_2(\underline{d}_1) + \delta_2(\underline{d}_1') + 1) \\ &\quad + \sum_{j=2}^{g} \delta_2(\underline{c}_j)\delta_2(\underline{d}_j) + \sum_{j=2}^{h} \delta_2(\underline{e}_j)\delta_2(\underline{f}_j) \\ &\equiv \delta_2(\underline{c}) + \mathrm{arf}(X) + \mathrm{arf}(Y) \mod 2\end{aligned}$$

and $\delta_2(\underline{c}) \equiv \beta(L) \pmod 2$ so,

$$\operatorname{arf}(L) \equiv \operatorname{arf}(X) + \operatorname{arf}(Y) + \beta(L) \pmod 2. \quad \square$$

Corollary 1. *The Arf invariant of a link of linking number zero is independent of the orientation of the components of the link.*

Proof. We must show that $\beta(L) \pmod 2$ is independent of the link's orientation.

Definition 1.5 for $\beta(L)$ shows

$$\begin{aligned}\beta(L) &= 2\sum_{i<j} \operatorname{lk}(c_i, c_j)L + \sum_{j=1}^{2p} \operatorname{lk}(c_j, c_j^+) \\ &\equiv \sum_{j=1}^{2p} \operatorname{lk}(c_j, c_j^+) \pmod 2.\end{aligned}$$

Each term $\operatorname{lk}(c_j, c_j^+)$ is independent of the orientation of the components of the link. $\square$

3. A formula for $|\beta(L)|$ in terms of the Alexander polynomial of L

In this section we establish a relationship between the $\underline{\beta}$ invariant of a link and its Alexander polynomial $\Delta(x, y)$. If $L = X \cup Y$ is a link of linking number zero then it has an Alexander polynomial of the form

$$\Delta(x, y) = (x-1)(y-1)f(x, y).$$

Theorem 2. $|\beta(L)| = |f(1, 1)|$.[1]

We compute $\Delta(x, y)$ using an algorithm by Cooper [Co] which is summarized in the following paragraph:

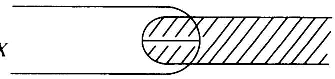
Let $L = X \cup Y$ be an oriented link in S^3 and M_X, M_Y Seifert surfaces with orientations inherited from those of X and Y such that $M_X \cap M_Y$ consists of only clasp intersections (see Figure 3.1(i)). Cooper defines two bilinear forms

$$u, v\colon H_1(M_X \cup M_Y) \times H_1(M_X \cup M_Y) \to Z$$

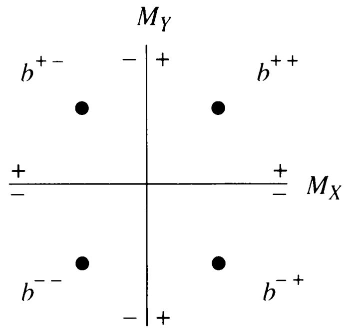
by $u(\underline{a}, \underline{b}) = \operatorname{lk}(a, b^{--})$ and $v(\underline{a}, \underline{b}) = \operatorname{lk}(a, b^{-+})$, where a, b are curves in M_X and M_Y respectively which represent $\underline{a}$ and $\underline{b}$. The notation $\operatorname{lk}(a, b^{--})$ stands for the linking number between a and b^{--} where b^{--} is the push off of b in the negative normal direction to M_X and negative normal direction to M_Y. Similarly $\operatorname{lk}(a, b^{-+})$ is the linking number between a and the push off of b in the negative normal direction to M_X and positive normal direction to M_Y. When a or b passes from M_X to M_Y it must contain the clasp intersection. Near a clasp the push off of a curve will lie in one of the four quadrants formed by the clasp intersection as suggested in Figure 3.1(ii). Cooper observed that

[1]The quantity $|f(1, 1)|$ is equal to $|a_1|$, where a_1 is a Conway polynomial coefficient. In [C] the Conway polynomial coefficients are found to be related to Milnor's μ-invariants and to certain linking numbers.

(i) A clasp intersection



(ii)



(iii) Changing a circle intersection to two clasps

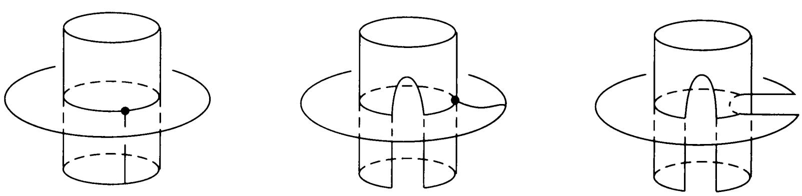


FIGURE 3.1

$$H_1(M_X \cup M_Y) \cong H_1(M_X) \oplus H_1(M_Y) \oplus \widehat{H}_0(M_X \cap M_Y).$$

Loops circling through the clasps form a basis for the component of $H_1(M_X \cup M_Y)$ isomorphic to $\widehat{H}_0(M_X \cap M_Y)$. Let A, B be matrices representing u and v respectively. Cooper shows that

$$\Delta(x, y) = (x-1)^{-2g}(y-1)^{-2h}\det(xyA + A^T - xB - yB^T),$$

where $g =$ genus of M_X and $h =$ genus of M_y.

Proof of Theorem 2. By Lemma 2.1 we can assume that we have oriented Seifert surfaces N'_X and N'_Y for X and Y such that $N'_X \cap N'_Y$ is a single circle. The circle intersection can be changed as in [Co] to two clasp intersections (see Figure 3.1(iii)). Let N_X, N_Y be the resulting Seifert surfaces for X and Y such that $N_X \cap N_Y$ is two clasps. If $N'_X \cap N'_Y = c$ then the invariant of L,

$\beta(L)$, is by definition equal to $\operatorname{lk}(c, c^{--}) = \operatorname{lk}(c, c^{-+})$. Let us choose a basis of $H_1(N_X \cup N_Y)$ consisting of a basis for $H_1(N_X)$ union a basis for $H_1(N_Y)$ union $\underline{c}$ where c runs through the two clasps as above and represents $\underline{c}$. Notice that if curve b lies in M_X then $u(\underline{a}, \underline{b}) = v(\underline{a}, \underline{b}) = \operatorname{lk}(a, b^-)$ where b^- is the push off of b in the negative normal direction to M_X. If $A_X = A$ restricted to $H_1(N_X)$ and $B_X = B$ restricted to $H_1(N_X)$ then $A_X = B_X$. Similarly if $A_Y = A$ restricted to $H_1(N_Y)$ and $B_Y = B$ restricted to $H_1(N_Y)$ then $B_Y = A_Y^T$. Let $n = \beta(L) = \operatorname{lk}(c, c^{--}) = \operatorname{lk}(c, c^{-+})$. Then with respect to the above basis for $H_1(N_X \cup N_Y)$, A and B have the following form:

$$A = \begin{bmatrix} A_X & & \\ & & q_{ij} \\ & A_y & \\ q_{ji} & & n \end{bmatrix}, \qquad B = \begin{bmatrix} A_X & & \\ & & q_{ij} \\ & A_Y^T & \\ q_{ji} & & n \end{bmatrix}$$

where $q_{ij} = q_{ji}$. A has matrices A_X, A_Y and (n) down the diagonal. Away from these submatrices A has ijth entry designated by q_{ij}. The q_{ij} entries represent linking numbers between nonintersecting curves in N_X and N_Y, thus $q_{ij} = q_{ji}$. B has matrices A_X, A_Y^T and (n) down the diagonal. The q_{ij} entries of B are equal to those of A.

By Cooper's algorithm

$$\begin{aligned}\Delta(x, y) &= (x-1)^{-2g}(y-1)^{-2h} \det xy \begin{bmatrix} A_X & & \\ & & q_{ij} \\ & A_Y & \\ q_{ji} & & n \end{bmatrix} \\ &\quad + \begin{bmatrix} A_X^T & & \\ & & q_{ji} \\ & A_Y^T & \\ q_{ij} & & n \end{bmatrix} - x \begin{bmatrix} A_X & & \\ & & q_{ij} \\ & A_Y^T & \\ q_{ji} & & n \end{bmatrix} - y \begin{bmatrix} A_X^T & & \\ & & q_{ij} \\ & A_Y & \\ q_{ji} & & n \end{bmatrix} \\ &= (x-1)^{-2g}(y-1)^{-2h} \\ &\quad \cdot \det \begin{bmatrix} (y-1)(xA_X - A_X^T) & & (x-1)(y-1)q_{ij} \\ & (x-1)(yA_Y - A_Y^T) & \\ (x-1)(y-1)q_{ji} & & \\ & & (x-1)(y-1)n \end{bmatrix}.\end{aligned}$$

Now factor out $(x-1)$ from the first $2g$ rows and $(y-1)$ from the next $2h$ rows:

$$= \det \begin{bmatrix} xA_X - A_X^T & \cdots & (x-1)q \\ (y-1)q \cdots yA_Y - A_Y^T & & (y-1)q \\ (x-1)(y-1)q & \cdots & (x-1)(y-1)n \end{bmatrix}.$$

Factor our $(x-1)(y-1)$ from the last row:

$$\begin{aligned}&= (x-1)(y-1) \det \begin{bmatrix} xA_X - A_X^T & & (x-1)q \\ (y-1)q \cdots y & A_Y - A_Y^T & (y-1)q \\ q & \cdots & n \end{bmatrix} \\ &= (x-1)(y-1) f(x, y).\end{aligned}$$

$$f(1, 1) = \det \begin{bmatrix} A_X - A_X^T & & 0 \\ & A_Y - A_Y^T & \vdots \\ q & \cdots \quad q & n \end{bmatrix}$$
$$= \det(A_X - A_X^T)\det(A_Y - A_Y^T)n = \pm n = \pm \underline{\beta}(L)$$

since $A_X - A_X^T$ and $A_Y - A_Y^T$ are intersection forms on $H_1(N_X)$ and $H_1(N_Y)$ and therefore have determinant ± 1. □

4. The β_j invariants

If $L = X \cup Y$ is a link of linking number zero then Cochran has defined a series of integral concordance invariants β_j, where $\beta_0 = \beta$. To define β_j we must first define the notion of derived links of a link.

Definition 4.1 (Cochran [C]). If $L = X \cup Y$ is a link of linking number zero and M_X, M_Y are Seifert surfaces spanning X and Y such that $M_X \cap M_Y = c_1$ where c_1 is an embedded circle, then $L = X \cup c_1$ is a *first derived link of* L. Link $L_j = X \cup c_j$ is a *jth derived link of* L if it is a first derived link of $L_{j-1} = X \cup \underline{c}_{j-1}$. (*Note*: Each derived link has the same first component.)

Definition 4.2 (Cochran [C]). The β_j invariant of L, $j > 0$, is equal to the β invariant of a jth derived link of L and $\beta_0 = \beta$.

Theorem 3. *Let $L = X \cup Y$ be a link of linking number zero where X is the unknot. Let Σ be the $Z/2$ cover of S^3 branched over X and Y^0, Y^1 the two lifts of Y in Σ. Then*

$$-\frac{1}{2}\operatorname{lk}(Y^0, Y^1) = \sum_{j=0}^{N-1} 4^j \beta_j(L) - 2 \circ 4^{N-1} \operatorname{lk}(c_N^0, c_N^1), \tag{1}$$

where $L_N = X \cup c_N$ is an Nth derived link of L, and c_N^0, c_N^1 are the two lifts of c_N in Σ.

Corollary 3. $\frac{1}{2}\operatorname{lk}(Y^0, Y^1) \equiv \beta(L) \mod 2$.

Proof. This is just line (1) modulo two. □

Proof of Theorem 5. Let M_X, M_Y be oriented Seifert surfaces for X and Y such that $M_X \cap M_Y = c_1$ where c_1 is an embedded circle. In Σ, M_Y (cut along c_1) lifts to M_Y^0 and M_Y^1; c_1 lifts to c_1^0 and c_1^1. Now,

$$\partial M_Y^0 = c_1^0 - c_1^1 + Y^0 \quad \text{and} \quad \partial M_Y^1 = c_1^1 - c_1^0 + Y^1 .$$

Curve c_1 has a neighborhood in M_Y parametrized by $c_1 \times [-1, 1]$. We will choose γ in M_Y to be equal to $c_1 \times -1$ or $c_1 \times 1$. The two lifts of γ will be contained in collar neighborhoods of ∂M_Y^0 and ∂M_Y^1. Choose γ so that γ^0, the lift of γ in M_Y^0, is contained in a neighborhood of c_1^0 in M_Y^0, then γ^1, the lift of γ in M_Y will be contained in a neighborhood of c_1^1 in M_Y^1.

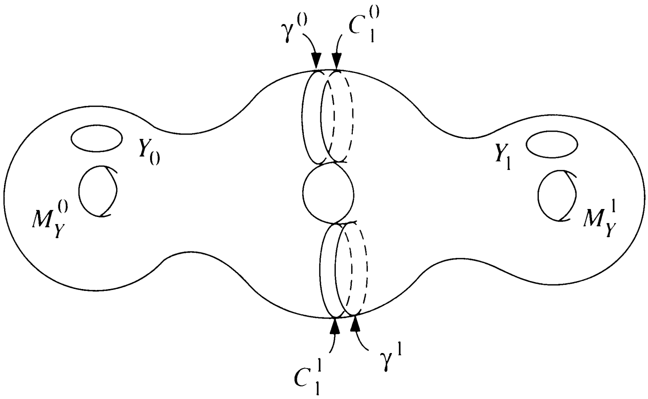


FIGURE 4.1

We would like to compute $\mathrm{lk}(Y^0, Y^1)$. First note that since X is the unknot Σ is homeomorphic to S^3 so the linking number is well defined. In $\Sigma - Y^1$, Y^0 is homologous to $c_1^1 - \gamma^0$ and Y^1 is homologous to $c_1^0 - \gamma^1$ (see Figure 4.1). So $\mathrm{lk}(Y^0, Y^1) = \mathrm{lk}(\gamma^0 - c_1^1, Y^1)$. In $\Sigma - (\gamma^0 - c_1^1)$, Y^1 is homologous to $\gamma^1 - c_1^0$, so

$$\begin{aligned}\mathrm{lk}(\gamma^0 - c_1^1, Y^1) &= \mathrm{lk}(\gamma^0 - c_1^1, \gamma^1 - c_1^0)\\ &= \mathrm{lk}(\gamma^0, \gamma^1) - \mathrm{lk}(\gamma^0, c_1^0) - \mathrm{lk}(c_1^1, \gamma^1) + \mathrm{lk}(c_1^1, c_1^0)\\ &= 2\,\mathrm{lk}(c_1^0, c_1^1) - 2\,\mathrm{lk}(\gamma^0, c_1^0)\end{aligned}$$

since $\mathrm{lk}(\gamma^0, \gamma^1) = \mathrm{lk}(c_1^0, c_1^1)$ and $\mathrm{lk}(\gamma^0, c_1^0) = \mathrm{lk}(c_1^1, \gamma^1)$.

We now use a general lifting formula of linking numbers: Let $p\colon \widetilde{W} \to W$ be a covering space projection, and G the group of covering transformations of $\widetilde{W}$. Assume linking numbers are well defined in W and $\widetilde{W}$. Let a, b be two closed curves in W. Then $\mathrm{lk}_W(a, b) = \sum_{\tau\in G} \mathrm{lk}_{\widetilde{W}}(\tau a, b)$. We have

$$\begin{aligned}\beta(L) &= \mathrm{lk}_{S^3}(c_1, c_1^+) = \mathrm{lk}_{S^3}(c_1, \gamma)\\ &= \mathrm{lk}(c_1^0, \gamma^0) + \mathrm{lk}(c_1^0, \gamma_1) = \mathrm{lk}(c_1^0, \gamma^0) + \mathrm{lk}(c_1^0, c_1^1).\end{aligned}$$

So

$$\begin{aligned}\mathrm{lk}(Y^0, Y^1) &= 2\,\mathrm{lk}(c_1^0, c_1^1) - 2\,\mathrm{lk}(\gamma^0, c_1^0)\\ &= 2\,\mathrm{lk}(c_1^0, c_1^1) - 2\,\mathrm{lk}(\gamma^0, c_1^0) - 2\,\mathrm{lk}(c_1^0, c_1^1) + 2\,\mathrm{lk}(c_1^0, c_1^1)\\ &= 4\,\mathrm{lk}(c_1^0, c_1^1) - 2[\mathrm{lk}(\gamma^0, c_1^0) + \mathrm{lk}(c_1^0, c_1^1)]\\ &= 4\,\mathrm{lk}(c_1^0, c_1^1) - 2\beta(L),\end{aligned}$$

and

$$\mathrm{lk}(Y^0, Y^1) = -2\beta(L) + 4\,\mathrm{lk}(c_1^0, c_1^1). \tag{2}$$

The above equation is a recursive one. Consider the link $L_1 = X \cup c_1$ a first derived link of L. By (2)

$$\operatorname{lk}(c_1^0, c_1^1) = -2\beta_1(L) + 4\operatorname{lk}(c_2^0, c_2^1)$$

where $L_2 = X \cup c_2$ is a second derived link of L. Thus,

$$\begin{aligned}\operatorname{lk}(Y^0, Y^1) &= -2\beta(L) + 4[-2\beta_1(L) + 4\operatorname{lk}(c_2^0, c_2^1)] \\ &= -2\beta(L) - 2\cdot 4\beta_1(L) + 4^2\operatorname{lk}(c_1^0, c_2^1).\end{aligned}$$

In general $\operatorname{lk}(c_j^0, c_j^1) = -2\beta_j(L) + 4\operatorname{lk}(c_{j+1}^0, c_{j+1}^1)$, so we have

$$\begin{aligned}\operatorname{lk}(Y^0, Y^1) = &-2\beta_0(L) - 2\cdot 4\beta_1(L) - 2\cdot 4^2\beta_2(L) \\ &- 2\cdot 4^3\beta_3(L) - \cdots + 4^N\operatorname{lk}(c_N^0, c_N^1)\end{aligned}$$

or

$$-\frac{1}{2}\operatorname{lk}(Y^0, Y^1) = \sum_{j=0}^{N-1}\beta_j(L) - 2\cdot 4^{N-1}\operatorname{lk}(c_N^0, c_N^1). \quad \square$$

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